Astronomy 405 (Spring 2013)
Homework 4 (due on Feb 15)

1. The Earth and Mars share many similar orbital characteristics, e.g., at the closest approach to the Sun during the summer months in the southern hemispheres, the spin axis tilts are both $\sim 24°$, etc. However, the Earth orbit has an eccentricity of $e = 0.0167$, while Mars has $e = 0.0935$.

   (a) At latitude = 24° south of the Earth, how much does the difference in distances to the sun affect the Sun's heating in summer and winter?

   (b) Repeat (a) for Mars.

   (c) At the same location on Earth, how much does the Sun's position in the sky affect the heating in summer and winter?

   (d) Repeat (c) for Mars.

   (e) Compare the results from (a) and (c) and discuss the cause of seasons on Earth.

   (f) Compare the results from (b) and (d) and discuss the cause of seasons on Mars.

   (g) Why does the southern polar cap grow to a larger size than the northern polar cap (both for winter months)?

2. Let's model the interior of Saturn. We start with the equation of hydrostatic equilibrium $\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$, where P is the pressure, M_r is the mass within radius r, and ρ is the density.

   (a) Assume that $P(r) = K\rho^2$, where K is a constant. Substitute this expression for the pressure in the hydrostatic equilibeium equation, clean up the equation, and differentiate both side of the equation to obtain the following differntial equation for density:
   $\frac{d^2\rho}{dr^2} + \frac{2}{r}\frac{d\rho}{dr} + (\frac{2\pi G}{K})\rho = 0.$

   (b) Show that by defining $k = (2\pi G/K)^{1/2}$, the density equation in (a) can be satisfied by $\rho(r) = \rho_c(\frac{\sin kr}{kr})$.

   (c) The density goes to zero on the surface of Saturn. For Saturn's radius ($\sim 9.4R_{\text{Earth}}$), what are the values of k and K? [Hint: $\sin\pi = 0$, so $kR = \pi$.]

   (d) $M_r = \int 4\pi r^2\rho dr$. Substitute the density expression into this mass equation and derive an expression of M_r in terms of r and ρ_c.
   [Hint: $\int r(\sin kr)dr = \frac{1}{k^2}\sin kr - \frac{r}{k}\cos kr$.]

   (e) Using Saturn's mass ($\sim 95M_{\text{Earth}}$) at the surface as the boundary condition, estimate Saturn's central density ρ_c.

   (f) Plot density as a function of radius.

   (g) Plot M_r as a function of radius.

3. According to the Virial theorem, in a gravitationally bound system, the total energy is equal to 1/2 the gravitational potential energy. For a uniform sphere, the gravitational potential energy is $-\frac{3}{5}\frac{GM^2}{R}$ and the total energy is $-\frac{3}{10}\frac{GM^2}{R}$.

   (a) What is the total energy of Jupiter now?

   (b) Before Jupiter condensed out of a gas cloud, the radius is much larger than the current radius. What was the initial total energy, if the radius was 1 AU?

   (c) Compare the initial energy in (b) and the current energy in (a). The difference is the amount of energy radiated away. How much energy has been radiated away?

   (d) If the energy has been radiate away at a constant rate over the last 4.6×10^9 yrs. What is the rate of energy output (in units of W)?

   (e) Compare the rate of energy output in (d) to the solar power absorbed by Jupiter (5×10^{17} W) and the current total power produced from the interior (3.35×10^{17} W). What does this comparison imply? (In other words, did Jupiter radiate more energy in the past or less?) How did this affect the evolution of its moons?

4. Saturn's Roche limit and rings. Saturn's mass is 95 $M_{\rm Earth}$ and its radius is 9.4 $R_{\rm Earth}$.

   (a) What is the average density of Saturn?

   (b) For an asteroid of density 2100 kg m^{-3}, how close to Saturn can it go before it is crushed by tidal force?

   (c) Repeat (b) for a comet of density 600 kg m^{-3}.

   (d) Compare the locations of Saturn's rings with these distacnes (Roche limits). Discuss the origin of Saturn's rings.