Astronomy 404
November 22, 2013

White Dwarfs (WDs)

The surface temperatures of WDs range from <8000 K to ~200,000 K. The coldest ones are discovered by large proper motion. The hottest ones are discovered by their UV and X-ray emission.

Central Conditions of WDs

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)\rho}{r^2} = -\frac{4}{3}\pi G\rho^2 r$$

$$P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$$

At $r = 0$,

$$P_c \approx \frac{2}{3}\pi G\rho^2 R_{\rm wd}^2 \approx 3.8\times 10^{22}\ {\rm N\ m^{-2}}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

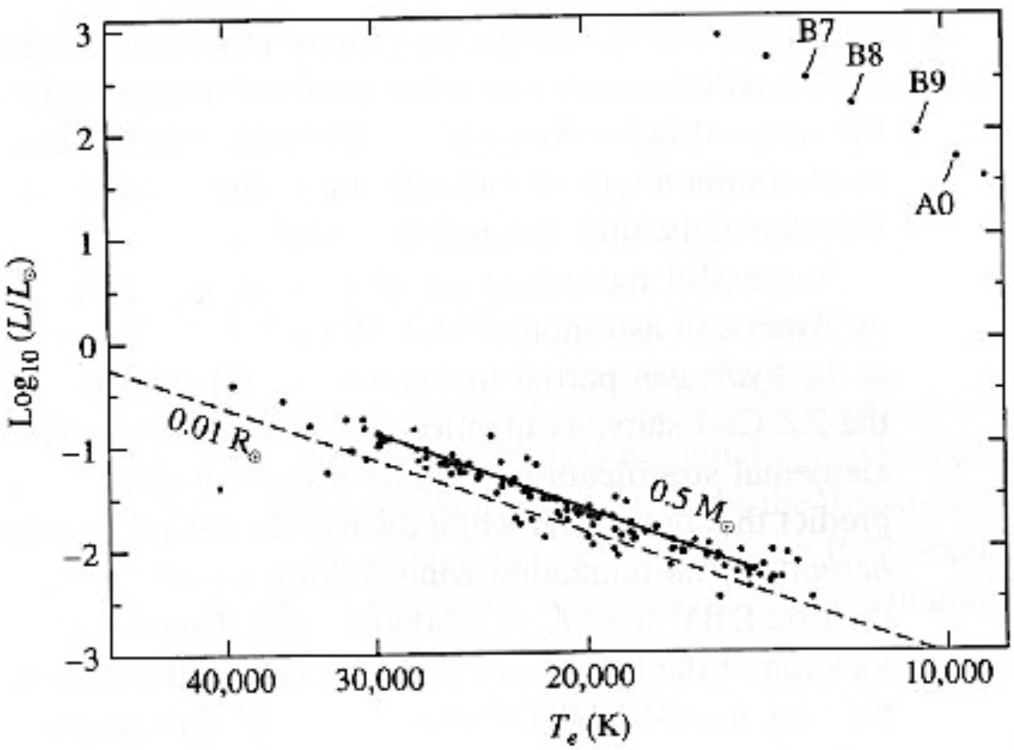
$$\frac{T_{\rm wd} - T_c}{R_{\rm wd} - 0} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T_c^3}\frac{L_{\rm wd}}{4\pi R_{\rm wd}^2}$$

$$T_c \approx \left[\frac{3\overline{\kappa}\rho}{4ac}\frac{L_{\rm wd}}{4\pi R_{\rm wd}}\right]^{1/4} \approx 7.6\times 10^7\ {\rm K}$$

for opacity of electron scattering = 0.02 $(1+X)$ $m^2\ kg^{-1}$, where X=0.

In the core of a WD, there cannot be any H, otherwise the high temperature and pressure would lead to violent burning. The core of a WD is dead, supported by electron degeneracy pressure.

WDs are the degenerate remnants of stars of initial masses up to ~8 $M_\odot$. Most DA WDs have masses between 0.42 and 0.7 $M_\odot$.



Above are DA WDs on an H-R diagram. The main sequence is at the upper right corner.

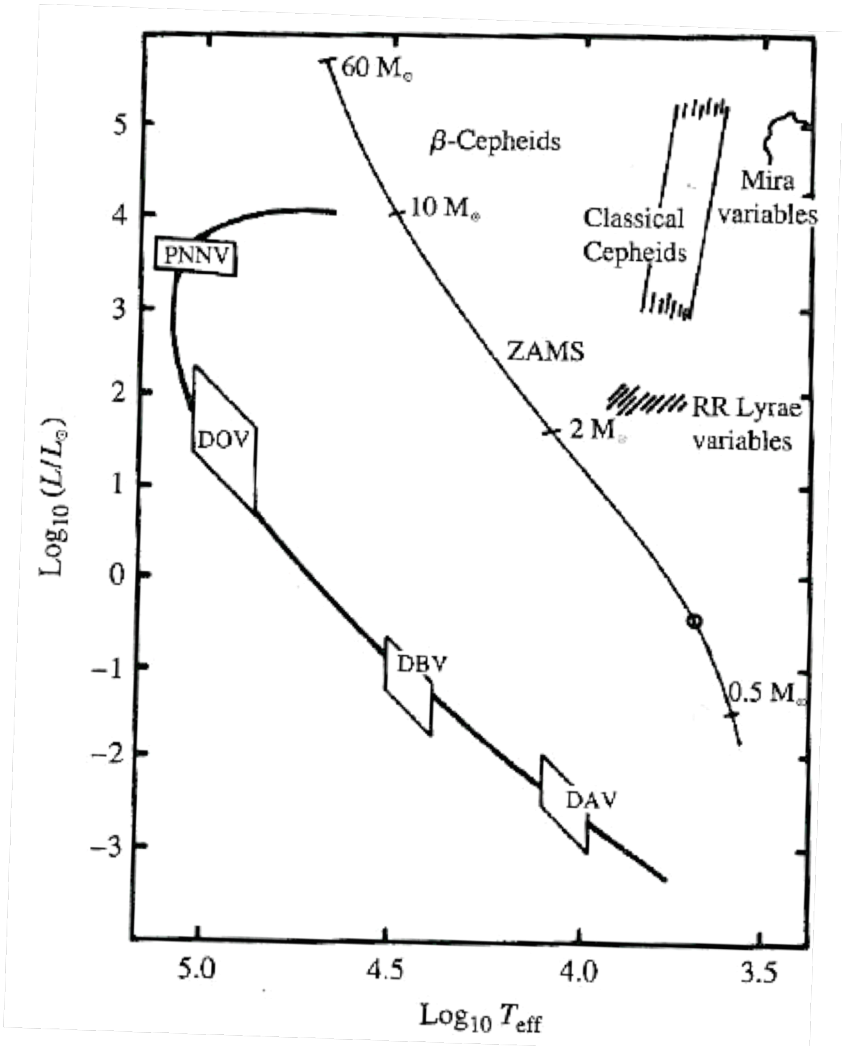
Surface Composition

DA WDs have a thin layer of H on top of a layer of He on top of a C-O core. This chemical differentiation takes place in ~100 year in the hot atmosphere of the star.

The origin of DB and DC WDs is not clear, could be due to convection, binarity, mass loss, or combinations of these.

DZ WDs are polluted by tidally pulverized asteroids.

Pulsating White Dwarfs



The pulsating WDs are driven by partial ionization zones.
DAV have $T_{\rm eff} \sim 12{,}000$ K -- hydrogen partial ionization
DBV have $T_{\rm eff} \sim 27{,}000$ K -- helium partial ionization
DOV have $T_{\rm eff} \sim 10^5$ K -- metal partial ionization

The pulsations correspond to non-radial *g*-modes that resonate in the surface layer of H and He.

The Physics of Degenerate Matter

Electrons are Fermions that follow Pauli's exclusion principle, so they can provide degeneracy pressure.
What are the quantum numbers of an electron when it is not bound to an atom?

At T = 0, if we have a completely **degenerate** gas, the electrons will occupy the lowest possible energy state (no excited state is occupied). The maximum energy that can be occupied is called the **Fermi Energy**.

Imagine a 3-D box of length L on each side, and think of electrons as standing waves in the box. The wavelengths in each dimension are:

$$\lambda_x = \frac{2L}{N_x}, \qquad \lambda_y = \frac{2L}{N_y}, \qquad \lambda_z = \frac{2L}{N_z},$$

Recall that $\lambda = h/p$ (de Broglie wavelength).

$$p_x = \frac{hN_x}{2L}, \qquad p_y = \frac{hN_y}{2L}, \qquad p_y = \frac{hN_x}{2L}.$$

The total kinetic energy is $\varepsilon = \frac{p^2}{2m}$, where $p^2 = p_x^2 + p_y^2 + p_z^2$.

Thus, $$\varepsilon = \frac{h^2}{8mL^2}(N_x^2 + N_y^2 + N_z^2) = \frac{h^2N^2}{8mL^2},$$

where $N^2 \equiv N_x^2 + N_y^2 + N_z^2$.

The total number of electrons within N is:

$$N_e = 2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi N^3\right)$$

where "2" is due to two spin quantum numbers, ±½; ⅛ is due to the requirement that all N_x, N_y, N_z are > 0 (the positive octant of *N*-space.

$$N = \left(\frac{3N_e}{\pi}\right)^{1/3}$$

Substituting *N* into the energy equation above,
we get the **Fermi Energy:**

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3},}$$

where *m* is the mass of the electron, and $n = N_e/L^3$.

The average energy per electron at zero temperature is $\frac{3}{5}\varepsilon_F$.

The Condition for Degeneracy

At temperature $T > 0$, some of the energy states higher than the Fermi energy are occupied and some lower energy states are vacant. All but the most energetic particles will have energy less than the Fermi energy.

We can express the Fermi energy in terms of density and temperature to see how the degree of degeneracy depends on density and temperature.

For full ionization, the number of electrons per unit volume is:

$$n_e = \left(\frac{\#\ \text{electrons}}{\text{nucleon}}\right)\left(\frac{\#\ \text{nucleons}}{\text{volume}}\right) = \left(\frac{Z}{A}\right)\frac{\rho}{m_H}$$

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3}$$

If the thermal energy of an electron is less than the Fermi energy, $\frac{3}{2}kT < \varepsilon_F$, it will not be able to occupy a higher energy level, and the electron gas is degenerate.

$$\frac{3}{2}kT < \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3}$$

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left[\frac{3\pi^2}{m_H}\left(\frac{Z}{A}\right)\right]^{2/3} = 1261\ \text{K m}^2\ \text{kg}^{-2/3}$$

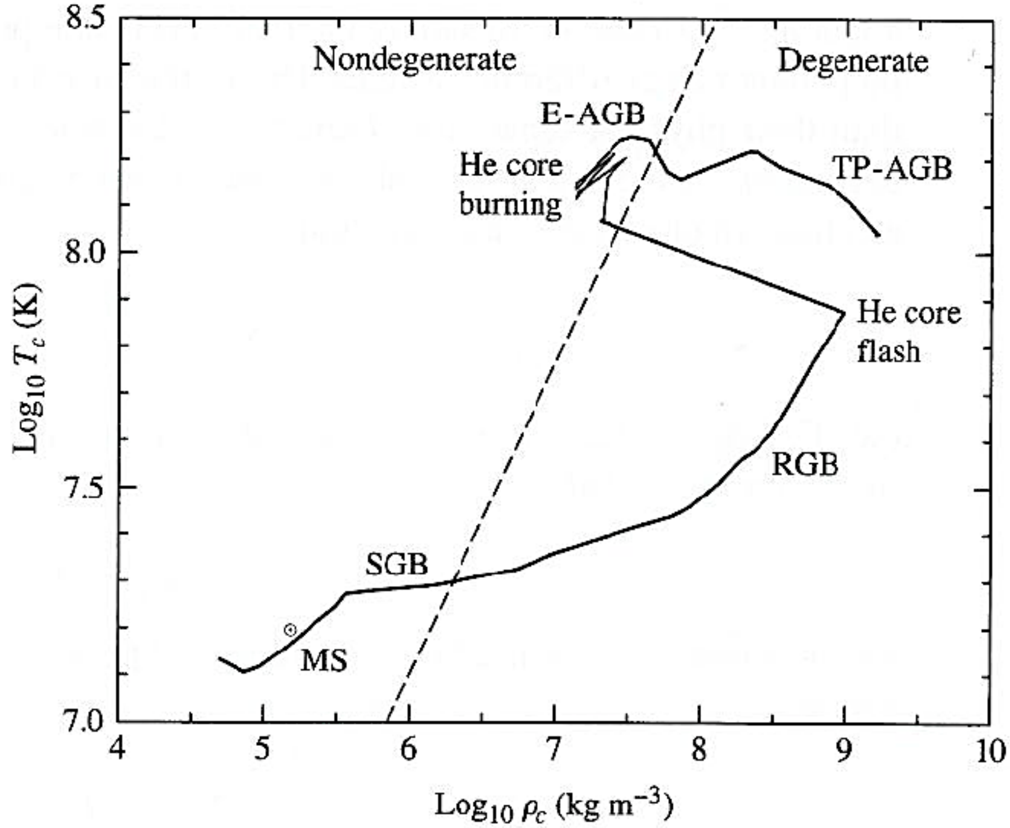
Defining $\mathcal{D} \equiv 1261\ \text{K m}^2\ \text{kg}^{-2/3}$, the condition for degeneracy can be written as:

$$\boxed{\frac{T}{\rho^{2/3}} < \mathcal{D}.}$$

At the center of the Sun, $T_c = 1.570 \times 10^7$ K, $\rho_c = 1.527 \times 10^5\ \text{kg m}^{-3}$

$$\frac{T_c}{\rho_c^{2/3}} = 5500\ \text{K m}^2\ \text{kg}^{-2/3} > \mathcal{D}.$$

At the center of Sirius B $\frac{T_c}{\rho_c^{2/3}} = 37\ \text{K m}^2\ \text{kg}^{-2/3} \ll \mathcal{D}$.



Degeneracy Pressure of Electrons

Two key ideas of quantum mechanics are used here:
1. Pauli's exclusion principle – one electron per quantum state,
2. Heisenberg's uncertain principle

$$\Delta x \, \Delta p_x \approx \hbar$$

Read pages 567-569 for the derivation of degeneracy pressure.

$$P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_e} \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{5/3}.$$

Using Z/A = 0.5 for a C-O WD, the electron degeneracy pressure in Sirius B is ~1.9×10^{22} N m^{-2}, within a factor of 2 of the estimate from hydrostatic equilibrium equation.

Electron degeneracy pressure is responsible for maintaining hydrostatic equilibrium in a white dwarf.

Chandrasekhar Limit

There is a maximum mass for WDs – the Chandrasekhar limit.

Setting the degeneracy pressure equal to the central pressure expected by hydrostatic equilibrium:

$$\frac{2}{3}\pi G\rho^2 R_{\rm wd}^2 = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}$$

$$R_{\rm wd} \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{G m_e M_{\rm wd}^{1/3}}\left[\left(\frac{Z}{A}\right)\frac{1}{m_H}\right]^{5/3}.$$

Mass-volume relation:

$$\boxed{M_{\rm wd} V_{\rm wd} = \text{constant.}}$$

Piling more mass into a WD only makes it smaller, but there is limit. The electrons have to mover faster, but they cannot move faster than the speed of light.

$$v \approx \frac{\hbar}{m_e} \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{1/3}$$

When the electron speed approaches the speed of light, the pressure becomes:

$$\boxed{P = \frac{(3\pi^2)^{1/3}}{4} \hbar c \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{4/3}}$$

A smallest departure from the hydrostatic equilibrium will cause the WD to collapse as electron degeneracy pressure fails.

Combining this pressure and $P_c \approx \frac{2}{3} \pi G \rho^2 R_{\rm wd}^2$, and eliminating the density, we can get Chandrasekhar limit:

$$M_{\rm Ch} \sim \frac{3\sqrt{2\pi}}{8} \left(\frac{\hbar c}{G} \right)^{3/2} \left[\left(\frac{Z}{A} \right) \frac{1}{m_H} \right]^2 = 0.44\ \mathrm{M}_\odot$$

A more rigorous derivation for Z/A = 0.5 gives
M_{Ch} = 1.44 $\mathrm{M}_\odot$, called the **Chandrasekhar limit**.