Astronomy 404
October 11, 2013

Energy generation rates

PP Chain

$$\epsilon_{pp} \simeq \epsilon'_{0,pp}\rho X^2 f_{pp}\psi_{pp}C_{pp}T_6^4$$

CNO Cycle

$$\epsilon_{\rm CNO} \simeq \epsilon'_{0,\rm CNO}\rho X X_{\rm CNO}T_6^{19.9}$$

Triple alpha reaction

$$\epsilon_{3\alpha} \simeq \epsilon'_{0,3\alpha}\rho^2 Y^3 f_{3\alpha}T_8^{41.0}$$

Why is N abundance enhanced on the surface and in the ejecta of massive stars? Why does CNO cycle cause a reduction in carbon and an enhancement in nitrogen?

$$
\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + e^{+} + \nu_e \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^{+} + \nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}.
\end{aligned}
$$

$$
\begin{aligned}
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{16}_{8}\mathrm{O} + \gamma \\
{}^{16}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{17}_{9}\mathrm{F} + \gamma \\
{}^{17}_{9}\mathrm{F} &\rightarrow {}^{17}_{8}\mathrm{O} + e^{+} + \nu_e \\
{}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He}.
\end{aligned}
$$

Initially, O > C > N. In the CNO cycle, the slowest reaction is fusing H with N (the bottleneck of the CNO cycle). The "traffic jam" in the CNO cycle virtually converts all initial C and N in the burning zone into ^{14}N. This is why we have WN stars, the nitrogen sequence of Wolf-Rayet stars.

Radiative transport of energy

$$\boxed{\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}.}$$

Convection transport of energy

Ideal gas law: $P = (\rho/\mu)\,kT$

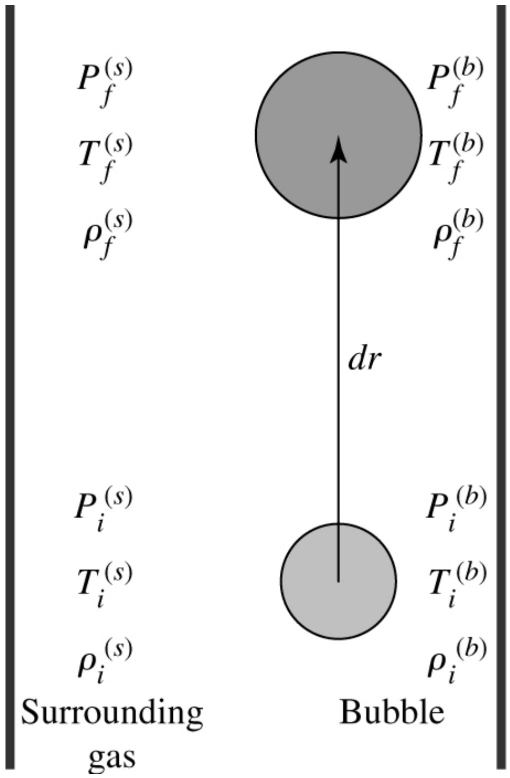
$$\frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Adiabatic gas law: $P\,V^{\gamma} = K$

$$\frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr}$$

$$\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\boxed{\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}.}$$



A Criterion for Stellar Convection

If the bubble has a lower density than the surroundings, it will rise. The buoyant force per unit volume exerted on the bubble is $f_B = \rho_i^{(s)} g$, and the downward gravitational force per unit volume is $f_g = \rho_i^{(b)} g$, so the net force per unit volume on the bubble is

$$f_{\text{net}} = -g\,\delta\rho$$

where $\delta\rho \equiv \rho_i^{(b)} - \rho_i^{(s)} < 0$.

The bubble rises adiabatically. If it is denser than the surroundings after rising a distance *dr*, it will sink and convection stops. If the surrounding is still denser than the bubble, it will keep on rising and convection results.

The gas is initially very nearly in thermal equilibrium, with $T_i^{(b)} \simeq T_i^{(s)}$ and $\rho_i^{(b)} \simeq \rho_i^{(s)}$. The bubble expands adiabatically in pressure equilibrium with the surroundings, $P_f^{(b)} = P_f^{(s)}$.

$$\rho_f^{(b)} \simeq \rho_i^{(b)} + \left.\frac{d\rho}{dr}\right|^{(b)} dr \quad \text{and} \quad \rho_f^{(s)} \simeq \rho_i^{(s)} + \left.\frac{d\rho}{dr}\right|^{(s)} dr.$$

The densities inside and outside the bubble are nearly equal, but need $\rho_f^{(b)} < \rho_f^{(s)}$ for the bubble to continue rising:

$$\left.\frac{d\rho}{dr}\right|^{(b)} < \left.\frac{d\rho}{dr}\right|^{(s)}$$

Recall that from the adiabatic gas law and ideal gas law:

$$\frac{dP}{dr} = \gamma \frac{P}{\rho} \frac{d\rho}{dr} \quad \text{and} \quad \frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Assuming that $d\mu/dr = 0$, the density gradient inequality equation above can be re-written as:

$$\frac{1}{\gamma}\frac{\rho_i^{(b)}}{P_i^{(b)}} \left.\frac{dP}{dr}\right|^{(b)} < \frac{\rho_i^{(s)}}{P_i^{(s)}} \left[\left.\frac{dP}{dr}\right|^{(s)} - \frac{P_i^{(s)}}{T_i^{(s)}} \left.\frac{dT}{dr}\right|^{(s)} \right]$$

Since $P^{(b)} = P^{(s)}$ all the time, it is necessary that

$$\left.\frac{dP}{dr}\right|^{(b)} = \left.\frac{dP}{dr}\right|^{(s)} = \frac{dP}{dr}$$

Drop the superscripts and cancelling the equivalent initial conditions:

$$\frac{1}{\gamma}\frac{dP}{dr} < \frac{dP}{dr} - \frac{P_i^{(s)}}{T_i^{(s)}}\left.\frac{dT}{dr}\right|^{(s)}$$

Dropping the subscripts and superscripts, we arrive at:

$$\left(\frac{1}{\gamma} - 1\right)\frac{dP}{dr} < -\frac{P}{T}\left.\frac{dT}{dr}\right|_{\text{act}}$$

$$\left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} > \left.\frac{dT}{dr}\right|_{\text{act}}$$

Recall that:

$$\left.\frac{dT}{dr}\right|_{\text{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

So, we have

$$\left.\frac{dT}{dr}\right|_{\text{ad}} > \left.\frac{dT}{dr}\right|_{\text{act}} \quad \text{or} \quad \left|\frac{dT}{dr}\right|_{\text{act}} > \left|\frac{dT}{dr}\right|_{\text{ad}}$$

$$\left(\frac{1}{\gamma}-1\right)\frac{dP}{dr} < -\frac{P}{T}\frac{dT}{dr}\bigg|_{act}$$

can be re-written as:

$$\frac{T}{P}\left(\frac{dT}{dr}\right)^{-1}\frac{dP}{dr} < -\frac{1}{\gamma^{-1}-1}$$

$$\frac{T}{P}\frac{dP}{dT} < \frac{\gamma}{\gamma-1}$$

$$\boxed{\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1}}$$

For an ideal monatomic gas, $\gamma = 5/3$, and convection occurs when $d \ln P \,/\, d \ln T < 2.5$.

Convection occurs when

(1) the stellar opacity is large, an unachievable $|dT/dr|_{act}$ is necessary for radiative transport

(2) a region where ionization is occurring, causing a large specific heat and a low $|dT/dr|_{ad}$

(3) temperature dependence of the nuclear energy generation rate is large, causing a steep radiative flux gradient and a large temperature gradient

(1) and (2) may occur in stellar atmosphere simultaneously.
(3) occurs in stellar interiors with CNO cycles or tripe alpha.

Mixing length

The mixing length is the scale length a bubble travels before giving up heat and merging into the surroundings. The ratio of the mixing length to the pressure scale height is a free parameter, usually ranges from 0.5 to 3.

There is a mixing length theory for approximation, but most modelers make assumptions and adjust free parameters in their codes to produce models that match observations.

Stellar Model Building

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \quad \text{(radiation)}$$

$$= -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} \quad \text{(adiabatic convection)}$$

Convection occurs when $\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma - 1}$, then the convection temperature gradient is purely adiabatic.

Bounary conditions

$M_r = 0$
$L_r = 0 \quad$ as $r \rightarrow 0$

$T \rightarrow 0$
$P \rightarrow 0$
$\rho \rightarrow 0 \quad$ as $r \rightarrow R_*$

Solve the equations numerically.

Polytropic Models and the Lane-Emden Equation

Before computers were widely available, astronomers did the best they can to solve these equations analytically. Assumptions have to be made, e.g.,

$P = K\rho^\gamma$ are known as **polytropes.**

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G(4\pi r^2\rho)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Remember Poisson's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_g}{dr}\right) = 4\pi G\rho$$

where $\Phi_g \equiv U_g/m$, is the gravitational potential energy per unit mass.

Applying $P = K\rho^\gamma$, we get

$$\frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho.$$

Define $\gamma \equiv (n+1)/n$, where *n* is the polytropic index; then we get:

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

$$\rho(r) \equiv \rho_c[D_n(r)]^n, \qquad \text{where } 0 \le D_n \le 1$$

$$\left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dD_n}{dr}\right]=-D_n^n.$$

Define:

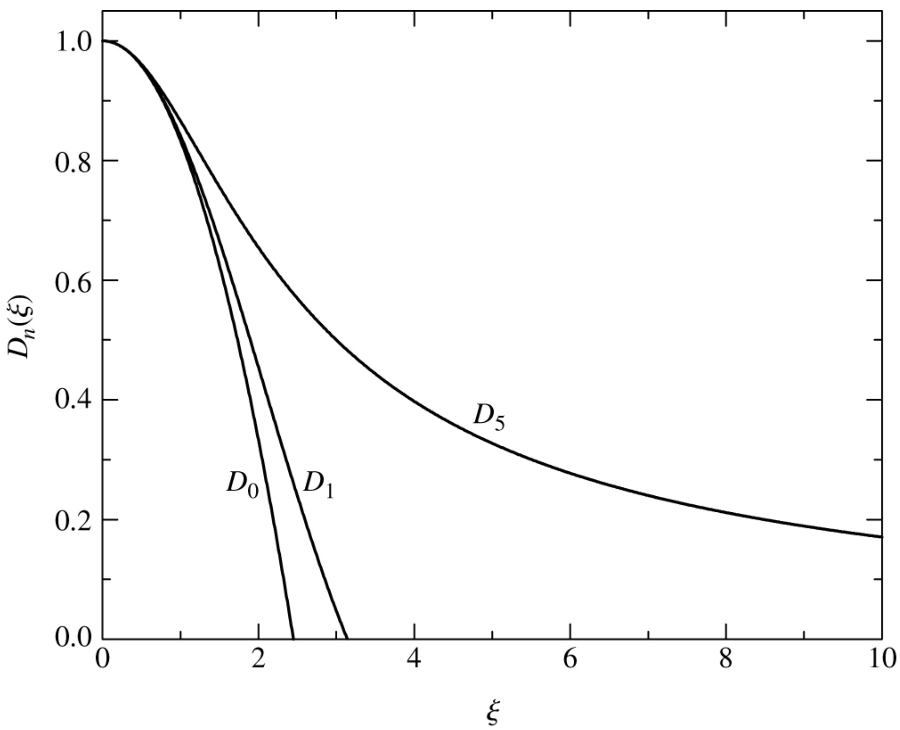
$$\lambda_n \equiv \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

and

$$r \equiv \lambda_n \xi$$

Finally, the **Lane-Emden equation:**

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_n}{d\xi}\right]=-D_n^n,}$$



D is a function for density, $r \equiv \lambda_n \xi$