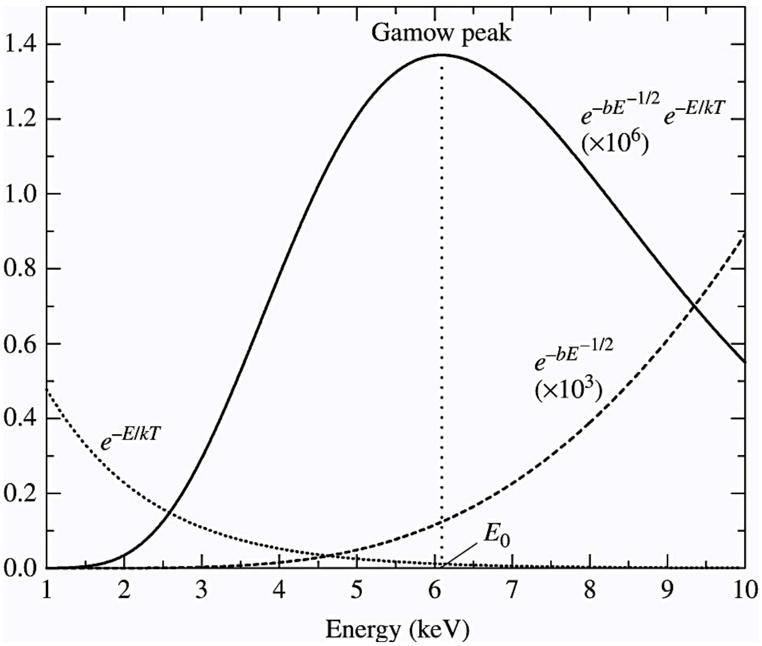
Astronomy 404
October 9, 2013

Nuclear reaction rate:

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E)\, e^{-bE^{-1/2}} e^{-E/kT}\, dE.$$

$e^{-bE^{-1/2}}$ from the tunneling increases with increasing E

$e^{-E/kT}$ from the velocity distrib. decreases with increasing E



The Gamow peak occurs at energy

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3}.$$

Energy generation rate is expressed as:

$$\epsilon_{ix} = \left(\frac{\mathcal{E}_0}{\rho}\right) r_{ix} \quad = \epsilon_0' X_i X_x \rho^\alpha T^\beta$$

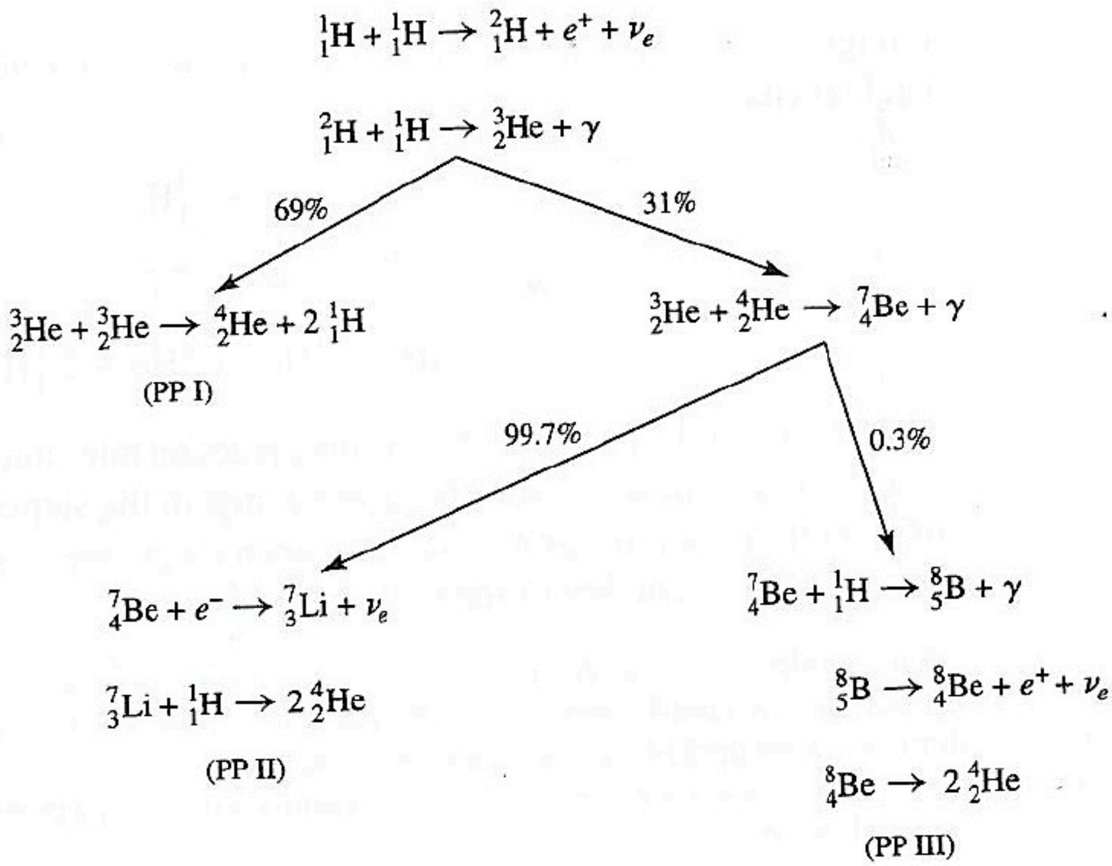
Luminosity Gradient Equation:

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon,}$$

Conservation laws for nuclear reaction:

electric charge, number of nucleons, number of leptons

Three branches of the PP Chain - PP I, PP II, PP III



Energy generation rate of PP chains:

$$\epsilon_{pp} = 0.241 \rho X^2 f_{pp} \psi_{pp} C_{pp} T_6^{-2/3} e^{-33.80 T_6^{-1/3}} \ \mathrm{W\,kg^{-1}}$$

where $T_6 \equiv T/10^6\,\mathrm{K}$ is the temperature,

$f_{pp} = f_{pp}(X, Y, \rho, T) \simeq 1$ is the screening factor,

$\psi_{pp} = \psi_{pp}(X, Y, T) \simeq 1$ is a correction factor of PP chains,

$C_{pp} \simeq 1$ involves higher-order correction terms.

Near $T = 1.5 \times 10^7$ K ($T_6 = 15$),

$$\epsilon_{pp} \simeq \epsilon'_{0,pp} \rho X^2 f_{pp} \psi_{pp} C_{pp} T_6^4$$

where $\epsilon'_{0,pp} = 1.08 \times 10^{-12}$ W m^3 kg^{-2}.

Remember that $X \equiv$ (mass of H)/(total mass).
$X = 0.70$ for typical young stars.

The CNO Cycle

In the CNO cycle, C, N, and O are used as catalysts, being consumed and regenerated during the process.

$$\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + e^{+} + \nu_e \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^{+} + \nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}.
\end{aligned}$$

About 0.04% of the time, the last reaction of the above CNO cycle produces ^{16}O and a photon, rather than ^{12}C and ^{4}He:

$$^{15}_{7}N + ^{1}_{1}H \rightarrow ^{16}_{8}O + \gamma$$
$$^{16}_{8}O + ^{1}_{1}H \rightarrow ^{17}_{9}F + \gamma$$
$$^{17}_{9}F \rightarrow ^{17}_{8}O + e^{+} + \nu_e$$
$$^{17}_{8}O + ^{1}_{1}H \rightarrow ^{14}_{7}N + ^{4}_{2}He.$$

The energy generation rate is:

$$\epsilon_{CNO} = 8.67 \times 10^{20} \rho X X_{CNO} C_{CNO} T_6^{-2/3} e^{-152.28 T_6^{-1/3}} \text{ W kg}^{-1}$$

Near $T = 1.5 \times 10^7$ K ($T_6 = 15$),

$$\epsilon_{CNO} \simeq \epsilon'_{0,CNO} \rho X X_{CNO} T_6^{19.9}$$

where $\epsilon'_{0,CNO} = 8.24 \times 10^{-31} \text{ W m}^3 \text{ kg}^{-2}$

Energy generation rate of PP chain is $\propto T^4$
Energy generation rate of CNO Cycle is $\propto T^{19.9}$

CNO cycle dominates in stars slightly more massive than the Sun, and PP chain dominates in less massive stars.

When H is converted into He, the mean molecular weight (μ) increases. For the same mass density, the number density of particles decreases, and thus the pressure decreases and the star collapses, raising the temperature and density to compensate the increase in μ.

When the temperature is high enough, helium begins to burn…

Triple Alpha Process of Helium Burning

$$^{4}_{2}\mathrm{He} + ^{4}_{2}\mathrm{He} \rightleftharpoons ^{8}_{4}\mathrm{Be}$$

$$^{8}_{4}\mathrm{Be} + ^{4}_{2}\mathrm{He} \rightarrow ^{12}_{6}\mathrm{C} + \gamma.$$

In the first reaction, ^{8}Be is unstable and will rapidly decay back to 2 He nuclei, unless it is fused with another He nucleus immediately. Therefore, the two reactions together can be thought of as a three-body interaction, and thus the reaction rate of the triple alpha process depends on $(\rho Y)^3$.

$$\epsilon_{3\alpha} = 50.9\rho^2 Y^3 T_8^{-3} f_{3\alpha} e^{-44.027 T_8^{-1}} \ \mathrm{W\ kg^{-1}}$$

where $T_8 \equiv T/10^8$ K and f is the screening factor.

At $T = 10^8$ K ($T_8 = 1$),

$$\epsilon_{3\alpha} \simeq \epsilon'_{0,3\alpha} \rho^2 Y^3 f_{3\alpha} T_8^{41.0}$$

A 10% increase in temperature would result in 1.1^{41} ~50 times increase in energy generation rate!!!

At high temperatures environment of He burning, other reactions can happen:

$$^{12}_{6}\mathrm{C} + ^{4}_{2}\mathrm{He} \rightarrow ^{16}_{8}\mathrm{O} + \gamma$$

$$^{16}_{8}\mathrm{O} + ^{4}_{2}\mathrm{He} \rightarrow ^{20}_{10}\mathrm{Ne} + \gamma$$

At sufficiently high temperatures in massive stars, more nuclear reactions with higher potential barriers can occur.

At 6×10^8 K, carbon burning:

$$
{}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow
\begin{cases}
{}^{16}_{8}\mathrm{O} + 2\,{}^{4}_{2}\mathrm{He} & *** \\
{}^{20}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} & \\
{}^{23}_{11}\mathrm{Na} + p^{+} & \\
{}^{23}_{12}\mathrm{Mg} + n & *** \\
{}^{24}_{12}\mathrm{Mg} + \gamma &
\end{cases}
$$

At 10^9 K, oxygen burning:

$$
{}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow
\begin{cases}
{}^{24}_{12}\mathrm{Mg} + 2\,{}^{4}_{2}\mathrm{He} & *** \\
{}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He} & \\
{}^{31}_{15}\mathrm{P} + p^{+} & \\
{}^{31}_{16}\mathrm{S} + n & \\
{}^{32}_{16}\mathrm{S} + \gamma &
\end{cases}
$$

*** mark reactions that are endothermic (absorbing energy) as opposed to exothermic (releasing energy)

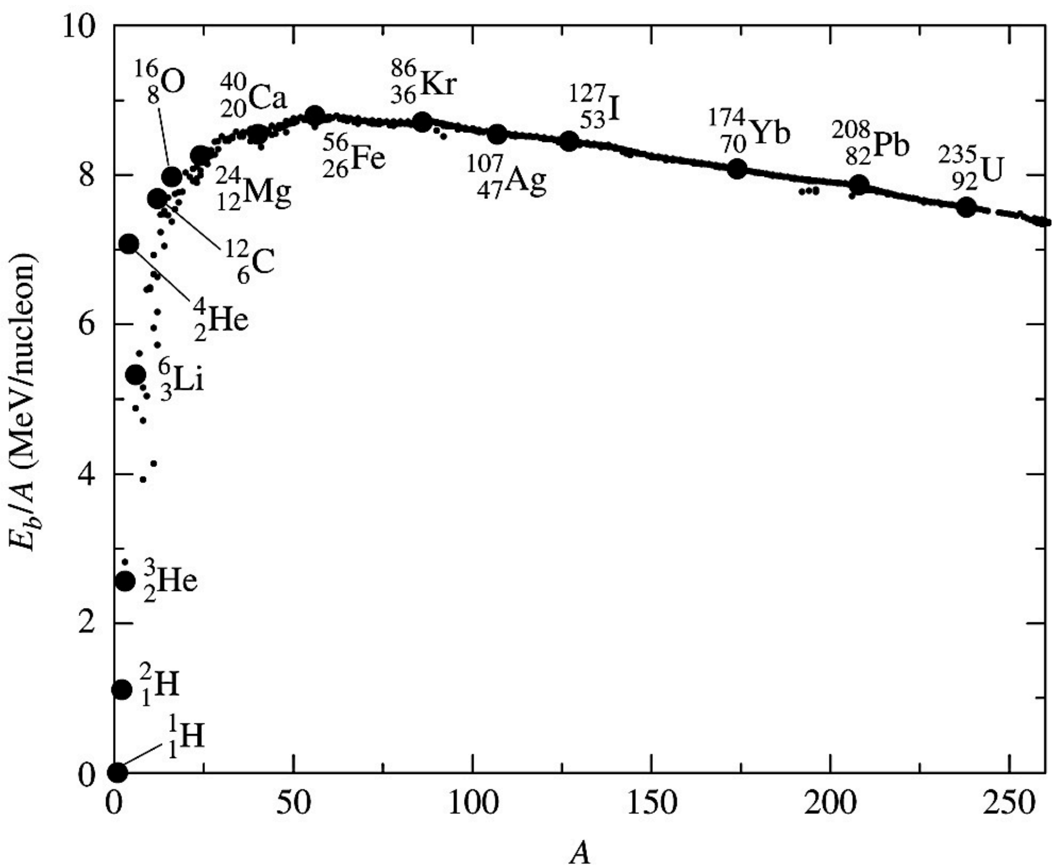
In general, endothermic reactions are much less likely to occur than exothermic reactions in stellar interiors.

Binding Energy Per Nucleon

Binding energy E_b :

$$E_b = \Delta mc^2 = \left[Zm_p + (A - Z)m_n - m_{\text{nucleus}}\right]c^2.$$

Binding energy per nucleon is E_b/A.



At the top of the peak is an isotope of iron, $^{56}_{26}Fe$.

H and He were produced in the Big Bang. Some Li, too.
The other elements were formed in stars then released.

The cosmic abundances, in order of H, He, O, C, Ne, N, Mg, Si, Fe, are results of stellar nucleosynthesis.

Energy Transport and Thermodynamics

We have had three fundamental equations to describe the stellar interior: *P, M, L* as functions of *r* . We'll find the temperature *T* as a function of *r* next.

Energies can be transported by:
(1) radiation,
(2) convection,
(3) conduction (generally not important in normal stars)

Radiative Transport of Energy

In order to make radiation flow, there has to be a radiation pressure gradient. As radiation pressure is dependent on temperature (Remember: TE or LTE), radiative transport of energy requires a temperature gradient.

$$\frac{dP_{\rm rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{\rm rad}$$

Recall that

$$P_{\rm rad} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\, d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4$$

$$F_{\rm rad} = \frac{L_r}{4\pi r^2}.$$

Substitute these two equations into the pressure gradient equation; then we get:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}.$$

The temperature gradient becomes steeper when opacity increases or density increases or temperature decreases.

Convection Transport of Energy

When the temperature gradient is too steep, convection may occur and transport energy outwards.

Consider a hot convective bubble of gas rises and expands *adiabatically*. After it travels a distance, it thermalizes with the surrounding gas, giving up its excess heat and blends in.

Ideal gas law: $P = (\rho/\mu)\, kT$

$$\frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Adiabatic gas law: $P\,V^{\gamma} = K$

where K is a constant, $\gamma \equiv C_P/C_V$, and the specific volume $V \equiv 1/\rho$.

So, $P = K\rho^{\gamma}$ and

$$\frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr}$$

Assuming a constant μ, substitute the pressure gradient back to the equation with a temperature gradient, then we get:

$$\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

Substituting the ideal gas law and hydrostatic equilibrium equation into this equation, we get:

$$\boxed{\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}.}$$

If the actual temperature gradient is greater than the adiabatic temperature gradient, the gas is **superadiabatic**.

$$\left|\frac{dT}{dr}\right|_{\mathrm{act}} > \left|\frac{dT}{dr}\right|_{\mathrm{ad}}$$

If the bubble expands and cool, but still hotter than the ambient gas, it will release heat, completing the mission of transporting energy outward.