Astronomy 404
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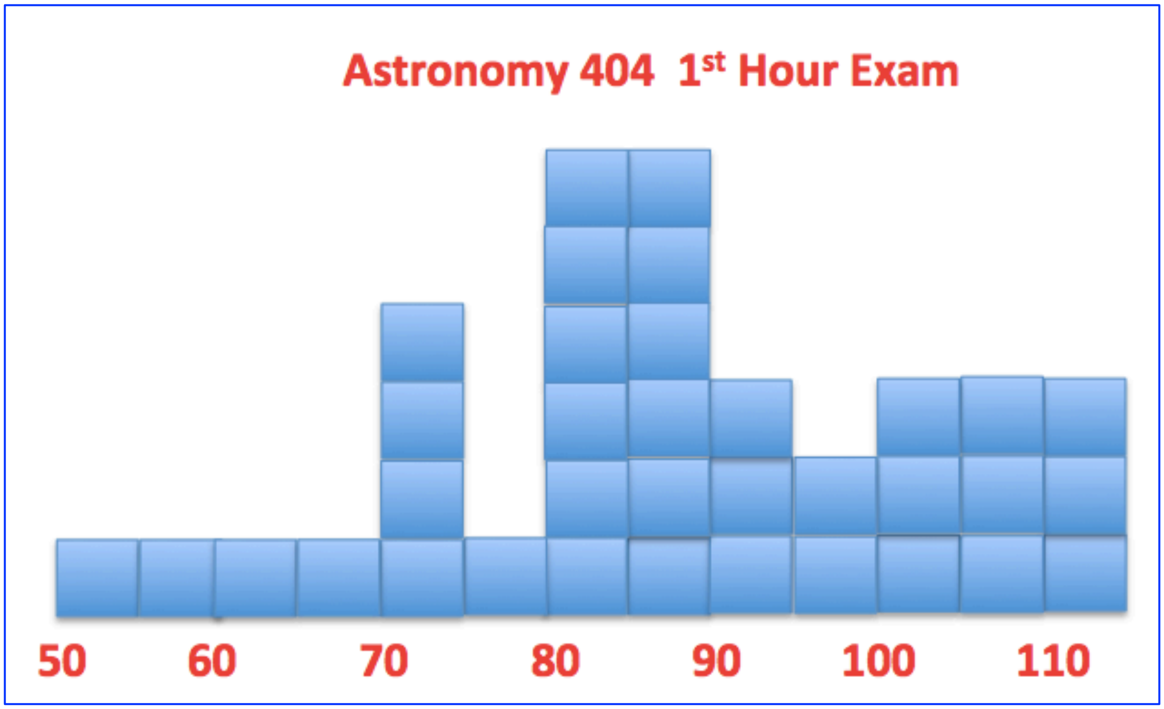


Chapter 10. The Interiors of Stars

Virial Theorem: $\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle$

Total energy of a uniform sphere: $E \sim -\frac{3}{10}\frac{GM^2}{R}$

Kelvin-Helmholtz timescale: $t_{KH} = \frac{\Delta E_g}{L_\odot}$

$E = mc^2$ **and** $1\ \mathrm{u} = 931.494\ \mathrm{MeV}/c^2$

Quantum Mechanical Tunneling across the potential barrier for nuclear reactions to start.

Nuclear Reaction Rates and the Gamow Peak

Now we know that the fusion of H into He can generate sufficient energy for the Sun to shine. Now we calculate the nuclear reaction rate…

We need to know two things:
(1) distribution of H nuclei as a function energy E
(2) probability for the tunneling to occur

Maxwell-Boltzmann distribution

$$n_v\,dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2\,dv.$$

Kinetic energy $K = E = \mu_m v^2/2$. Therefore,

$$n_E\,dE = \frac{2n}{\pi^{1/2}}\frac{1}{(kT)^{3/2}}E^{1/2}e^{-E/kT}\,dE$$

The probability for a reaction to happen is called a *cross section* :

$$\sigma(E) \equiv \frac{\text{number of reactions/nucleus/time}}{\text{number of incident particles/area/time}}$$

Consider the number of particles that will hit a target of cross section $\sigma(E)$. x denotes the target and i denotes an incident particle. The number of incident particle with energies between E and $E+dE$ is dN_E :

$$dN_E = \sigma(E)v(E)n_{iE}\,dE\,dt.$$

$$n_{iE}\, dE = \frac{n_i}{n} n_E\, dE,$$

where $n_i = \int_0^\infty n_{iE}\, dE, n = \int_0^\infty n_E\, dE$.

The number of reactions per target nucleus per time interval *dt* with energies between *E* and *E*+*dE* is

$$\frac{\text{reactions per nucleus}}{\text{time interval}} = \frac{dN_E}{dt} = \sigma(E) v(E) \frac{n_i}{n} n_E\, dE.$$

There are n_x targets per unit volume, so the total number of reactions per unit volume per unit time, over all energies, is

$$r_{ix} = \int_0^\infty n_x n_i \sigma(E) v(E) \frac{n_E}{n}\, dE$$

The cross section depends on two things:
(1) the size of a nucleus, and
(2) the tunneling probability

The size of the nucleus is approximately 1 de Broglie wavelength in radius ($r \sim \lambda$).

$$\sigma(E) \propto \pi \lambda^2 \propto \pi \left(\frac{h}{p}\right)^2 \propto \frac{1}{E}$$

The cross section for tunneling depends on the ratio of kinetic energy (E) to the barrier height (U_c) :

$$\sigma(E) \propto e^{-2\pi^2 U_c/E}.$$

Assuming $r \sim \lambda = h/p$,

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2/4\pi\epsilon_0 r}{\mu_m v^2/2} = \frac{Z_1 Z_2 e^2}{2\pi\epsilon_0 h v}$$

$$\sigma(E) \propto e^{-bE^{-1/2}},$$ where $$b \equiv \frac{\pi \mu_m^{1/2} Z_1 Z_2 e^2}{2^{1/2}\epsilon_0 h}.$$

Combining these two considerations, we get

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}.$$

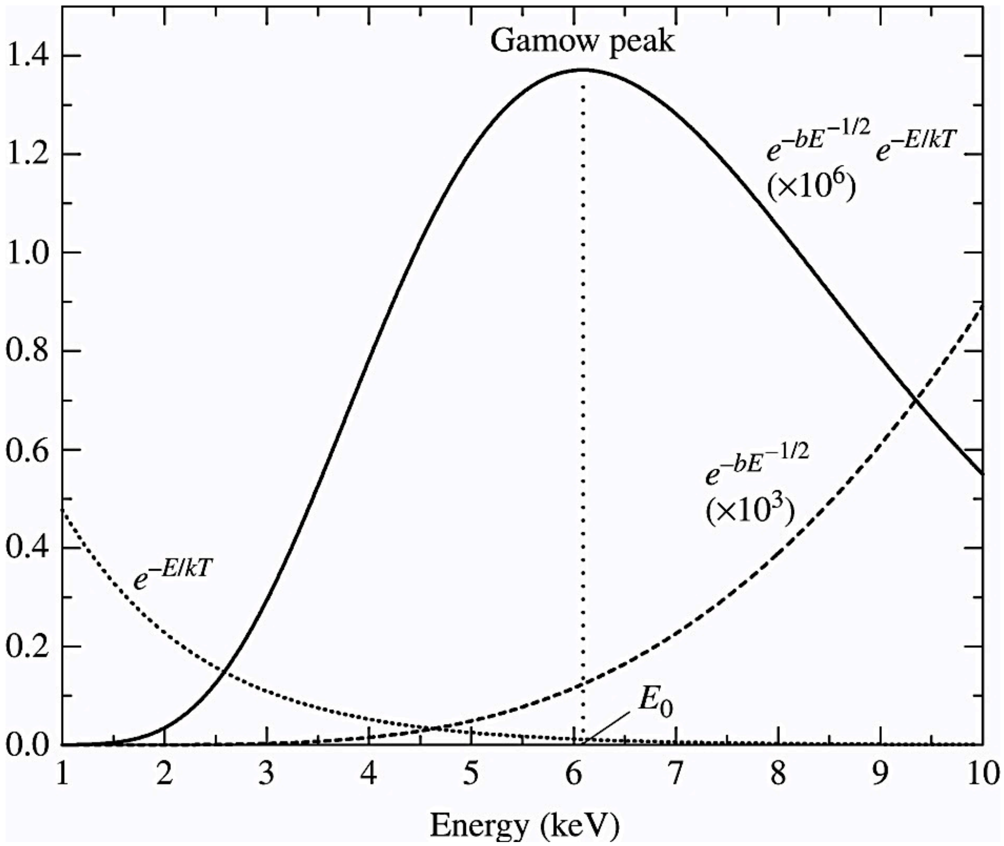
Substituting this equation and

$$n_E\, dE = \frac{2n}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT}\, dE$$ into $$r_{ix} = \int_0^\infty n_x n_i \sigma(E) v(E) \frac{n_E}{n}\, dE$$

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E)\, e^{-bE^{-1/2}} e^{-E/kT}\, dE.$$

$e^{-bE^{-1/2}}$ from the tunneling increases with increasing E

$e^{-E/kT}$ from the velocity distrib. decreases with increasing E



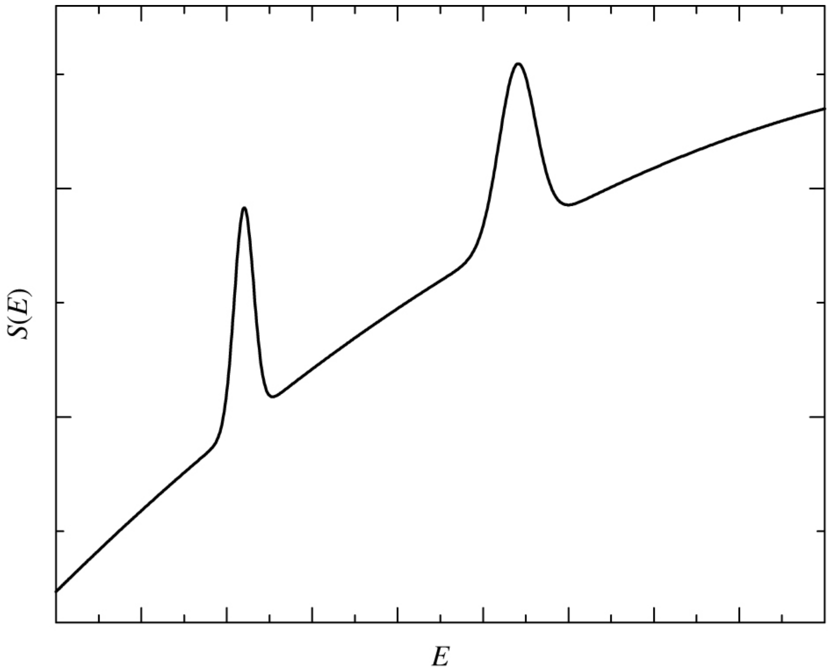
The Gamow peak occurs at the energy

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3}.$$

Because of the Gamow peak, the greatest contribution to the reaction rate comes in a fairly narrow energy band that depends on the temperature of the gas and the charges and masses of the constituents of the reaction.

$S(E)$ is a slow varying function and may be approximates as a constant $\rightarrow$ $S(E) \sim S(E_0)$ = constant.

Sometimes $S(E)$ may vary rapidly, peaking at specific energies, corresponding to energy levels within the nucleus: **resonance**.



Electron Screening

Electrons can partially hide the target nucleus, reducing its effective positively charge. The effective Coulomb potential barrier becomes

$$U_{\rm eff} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} + U_s(r)$$

where $U_s(r) < 0$ is the electron screening contribution. Electron screening can be significant, sometimes enhancing the He-producing reactions by 10% to 50%.

Representing Nuclear Reaction Rates Using Power Laws

$$r_{ix} \simeq r_0 X_i X_x \rho^{\alpha'} T^{\beta}$$

where r_0 is a constant, X_i and X_x are mass fractions of the two particles. $\alpha' = 2$ for two-body collisions, and β can range form 1 to 40 or more.

Combining the reaction rate (r_{ix}) and the energy released per reaction $\mathcal{E}_0$, the amount of energy liberated kg^{-1} s^{-1} becomes

$$\epsilon_{ix} = \left(\frac{\mathcal{E}_0}{\rho}\right) r_{ix} = \epsilon_0' X_i X_x \rho^\alpha T^\beta$$

in units of W kg^{-1}.

The sum of ϵ_{ix} for all reactions is the total energy generation rate.

Now we can talk about luminosity of the star…

$$dL = \epsilon\, dm$$

$$\epsilon = \epsilon_{\text{nuclear}} + \epsilon_{\text{gravity}}$$

As $dm = 4\pi r^2 \rho\, dr$, we get the **Luminosity Gradient Equation:**

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon,}$$

Remember the other fundamental equations?
$dp/dr = -\rho g$ hydrostatic equilibrium equation
$dm/dr = 4\pi r^2 \rho$ conservation of mass equation

Stellar Nucleosynthesis and Conservation Laws

Nucleosynthesis – the sequence of steps by which one element is converted into another.

The following items need to be conserved in nuclear reactions:
(1) electric charge
(2) number of nucleons
(3) number of leptons

Leptons are "light things" and include electrons, positrons, neutrinos and antineutrinos.

Matter and anti-matter annihilate. For example,

$$e^- + e^+ \rightarrow 2\gamma$$

Two photons are required to conserve both momentum and energy.

Neutrinos are electrically neutral and have a very small but non-zero mass $(m_\nu < 2.2\,\mathrm{eV}/c^2)$.

The cross section of neutrino is 10^{-48} m^2. In stellar interiors, the mean free path of a neutrino is $\sim 10^{18}$ m $\sim$ 10 pc. Neutrinos escape from the stellar interior easily.

The total number of matter leptons minus the total number of antimatter leptons must remain constant in a nuclear reaction.

Nuclei are represented by the symbol:

$$^{A}_{Z}\mathrm{X}$$

where X is the chemical symbol of the element, *A* is the mass number, and *Z* the atomic number.

The Proton-Proton Chains

First proton-proton chain (PP I):

$$4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + 2e^{+} + 2\nu_e + 2\gamma$$

(Check the conservation laws.)

The entire **PP I reaction chain** is:

$${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e$$

$${}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He} + \gamma$$

$${}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H}.$$

The first step is the slowest because it involves the decay of p^{+}, $p^{+} \rightarrow n + e^{+} + \nu_e$. Such a decay involves the weak force.

69% of the time ^{3}He interacts with another ^{3}He. The other 31% of time, ^{3}He interacts with ^{4}He. **PP II chain**:

$${}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{7}_{4}\mathrm{Be} + \gamma$$

$${}^{7}_{4}\mathrm{Be} + e^{-} \rightarrow {}^{7}_{3}\mathrm{Li} + \nu_e$$

$${}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \rightarrow 2\,{}^{4}_{2}\mathrm{He}.$$

^{7}Be can capture a proton and continue with **PP III chain**:

$$^{7}_{4}\text{Be} + {}^{1}_{1}\text{H} \rightarrow {}^{8}_{5}\text{B} + \gamma$$

$$^{8}_{5}\text{B} \rightarrow {}^{8}_{4}\text{Be} + e^{+} + \nu_e$$

$$^{8}_{4}\text{Be} \rightarrow 2\,{}^{4}_{2}\text{He}.$$

The three branches of PP chain and their branching ratios: