Astronomy 404
October 2, 2013

Chapter 10. The Interiors of Stars

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2} = -\rho g,$$

Equation of mass conservation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho,$$

Ideal gas law:

$$P_g = \frac{\rho k T}{\mu m_H}.$$

For neutral gas:

$$\frac{1}{\mu_n} = \sum_j \frac{1}{A_j} X_j \quad \simeq X + \frac{1}{4}Y + \left\langle \frac{1}{A} \right\rangle_n Z$$

, where $\langle 1/A \rangle_n \sim 1/15.5$

For ionized gas:

$$\frac{1}{\mu_i} = \sum_j \frac{1+z_j}{A_j} X_j \quad \simeq 2X + \frac{3}{4}Y + \left\langle \frac{1+z}{A} \right\rangle_i Z \quad \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

Total pressure = gas pressure + radiation pressure:

$$P_t = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$$

Stellar Energy Sources

Gravitational Enegy

Gravitational potential energy is negative:

$$U = -G\frac{Mm}{r}$$

Stars may contract and release gravitational potential energy, but not all energy released is available to be radiated away.

Virial Theorem: (Read pages 50-52 of Carroll & Ostlie)

For gravitationally bound systems in equilibrium, the total energy is ½ of the time-averaged potential energy: $\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle$.

(in other words: $-2\langle K \rangle = \langle U \rangle$)

The gravitational potential energy of *dm* at radius *r* is

$$dU_{g,i} = -G\frac{M_r\, dm_i}{r}$$

$$dm = 4\pi r^2 \rho\, dr$$

$$dU_g = -G\frac{M_r 4\pi r^2 \rho}{r}\, dr$$

$$U_g = -4\pi G \int_0^R M_r \rho r\, dr$$

We need to know $\rho\,(r)$ in order to do the integration, but we can use the average density as an approximation…

$$\rho \sim \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$M_r \sim \frac{4}{3}\pi r^3 \bar{\rho}.$$

$$U_g \sim -\frac{16\pi^2}{15} G\bar{\rho}^2 R^5 \sim -\frac{3}{5}\frac{GM^2}{R}$$

Using the Virial Theorem, $\langle E \rangle = \frac{1}{2}\langle U \rangle$, the total energy becomes:

$$E \sim -\frac{3}{10}\frac{GM^2}{R}$$

The Sun has a mass of 2×10^{30} kg and a radius of 6.96×10^{8} m. Its initial size is much larger than the current size. The energy radiated away during the collapse is:

$$\Delta E_g = -\left(E_f - E_i\right) \simeq -E_f \simeq \frac{3}{10}\frac{GM_\odot^2}{R_\odot} \simeq 1.1 \times 10^{41}\ \text{J}.$$

If the Sun's luminosity has been constant throughout its life, its age would be approximately:

$$t_{\text{KH}} = \frac{\Delta E_g}{L_\odot} \sim 10^7\ \text{yr}$$

t_{KH} is the Kelvin-Helmholtz timescale.

Moon rocks are over 4Gyr old. The Sun can't be just 10^7 yr old.

Chemical processes involve electron orbits and the relevant energy changes are in the 1-10 eV range.

Nuclear processes involve nucleons and the relevant energy ranges are MeV or higher.

Atomic number = Z = number of protons in the nucleus
Isotopes have the same Z, but different number of neutrons (N).
Mass number = $Z + N$ = number of nuceons

The masses of the proton, neutron, and electron are:

$$m_p = 1.67262158 \times 10^{-27} \text{ kg} = 1.00727646688 \text{ u}$$

$$m_n = 1.67492716 \times 10^{-27} \text{ kg} = 1.00866491578 \text{ u}$$

$$m_e = 9.10938188 \times 10^{-31} \text{ kg} = 0.0005485799110 \text{ u.}$$

u - tomic mass unit, and 1 u = $1.66053873 \times 10^{-27}$ kg, exactly 1/12 of the mass of the isotope carbon-12.

Using $E = mc^2$, we find 1 u = 931.494 MeV/c^2

$m_p + m_e - 13.6\text{eV} = m_H$ 13.6 eV - ionization potential of H

4 H nuclei can be fused into a He nucleus, called **fusion**.
$4\, m_p - m_{He} = 4 \times 1.0072765 - 4.002603$ u = 0.028697 u = 0.7% m_{He}
The amount of energy released is $E_b = \Delta mc^2 = 26.731$ MeV, the **binding energy** of the helium nucleus.

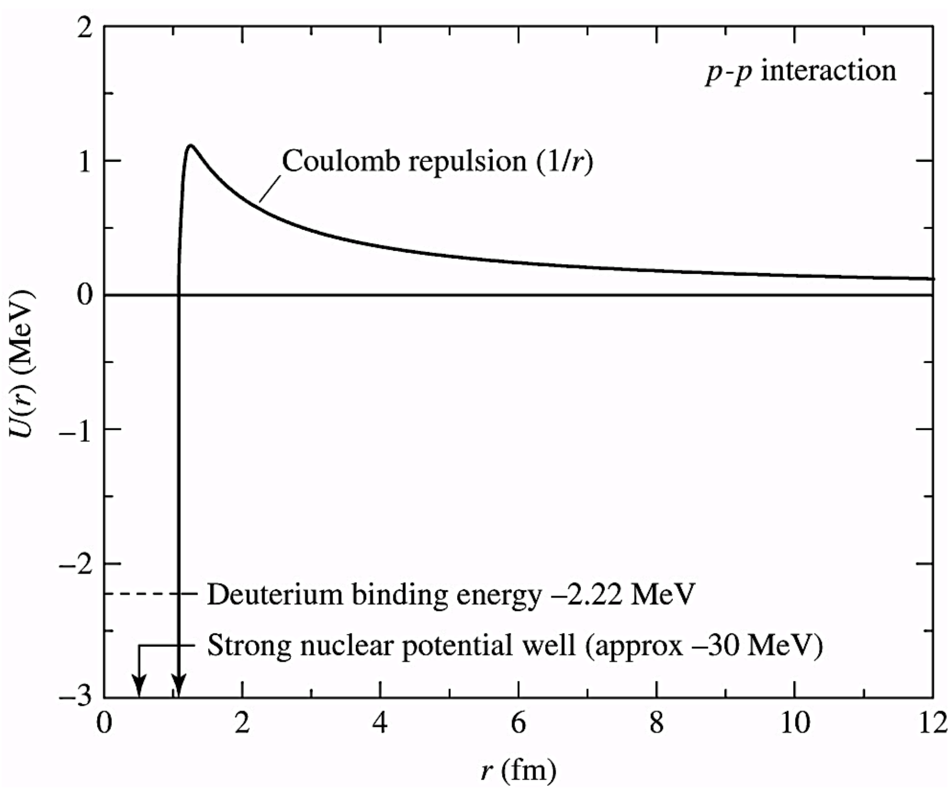
It takes 26.731 MeV to break apart a He nucleus.

Suppose 10% of the Sun's hydrogen mass can go through fusion into helium to generate energy. The amount of energy available is:

$$E_{\mathrm{nuclear}} = 0.1 \times 0.007 \times M_{\odot}c^2 = 1.3 \times 10^{44}\ \mathrm{J}$$

The **nuclear timescale** is $t_{\mathrm{nuclear}} = E_{\mathrm{nuclear}} / L_{\odot} \sim 10^{10}$ yr, more than adequate to account for the age of Moon rocks.

There is clearly adequate amount of nuclear energy in stars, but how do we get protons close enough for the strong nuclear force to take over?



1 fm = 10^{-15} m

If we assume that the Coulomb barrier is overcome by the thermal energy of the gas, the temperature $T_{\text{classical}}$ required is:

$$\frac{1}{2}\mu_m \overline{v^2} = \frac{3}{2}kT_{\text{classical}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$$

$$T_{\text{classical}} = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k r} \quad \sim 10^{10}\ \text{K}$$

μ_m is the reduced mass, = 0.5 m_p .
$Z_1 = Z_2 = 1$, and the radius r is of the order of 1 fm = 10^{-15} m.

At the center of the Sun, we do not have such a high temperature. In fact, the temperature is only ~1.57×10^7 K. Even with Maxwell-Boltzmann's high-velocity tail, there aren't enough energetic particles to overcome the Coulomb barrier to produce the Sun's observed luminosity.

Heisenberg's uncertainty principle make it possible to *cheat*...

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

$$\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m}$$

$$\frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m}$$

$$T_{\text{quantum}} = \frac{Z_1^2 Z_2^2 e^4 \mu_m}{12\pi^2\epsilon_0^2 h^2 k} \quad \sim 10^7\ \text{K}$$