Astronomy 404
September 30, 2013

Optical Depth & Opacity

$$I_\lambda(s) = I_{\lambda,0}\, e^{-\kappa_\lambda \rho s} + S_\lambda (1 - e^{-\kappa_\lambda \rho s})$$

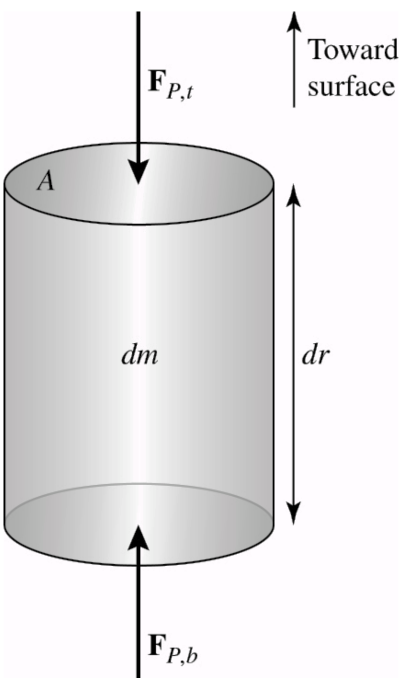
Chapter 10. The Interiors of Stars

Topics to be covered:
Hydrostatic Equilibrium
Pressure Equation of State
Stellar Energy Sources
Energy Transport and Thermodynamics
Stellar Model Building
The Main Sequence

Hydrostatic Equilibrium

The interior of a star cannot be observed directly.
(Only neutrinos can escape from the stellar interiors.)

A star is held together by gravity. The gravitational force is always attractive and the stellar mass tries to collapse. There must be an opposing force to prevent the star from collapsing.



Consider a cylinder of mass *dm* whose base is at distance *r* from the center of a spherical star.

Assume that the only forces acting on the cylinder are gravity and pressure force.

$\mathbf{F} = m\mathbf{a}$ (Newton's second law)

$$dm \frac{d^2r}{dt^2} = F_g + F_{P,t} + F_{P,b}$$

The force on the top is directed toward the star's center, $F_{P,t} < 0$, whereas the force on the bottom is directed outward, $F_{P,b} > 0$.

$$F_{P,t} = -\left(F_{P,b} + dF_P\right)$$

$$dm \frac{d^2 r}{dt^2} = F_g - dF_P$$

$$F_g = -G \frac{M_r \, dm}{r^2}$$

where M_r is the mass inside the sphere of radius r.

$$P \equiv \frac{F}{A} \quad \rightarrow \quad dF_P = A \, dP$$

$$dm \frac{d^2 r}{dt^2} = -G \frac{M_r \, dm}{r^2} - A \, dP$$

$$dm = \rho A \, dr$$

$$\rho A \, dr \frac{d^2 r}{dt^2} = -G \frac{M_r \rho A \, dr}{r^2} - A \, dP$$

$$\rho \frac{d^2 r}{dt^2} = -G \frac{M_r \rho}{r^2} - \frac{dP}{dr}$$

If the star is static, the acceleration must be zero; therefore,

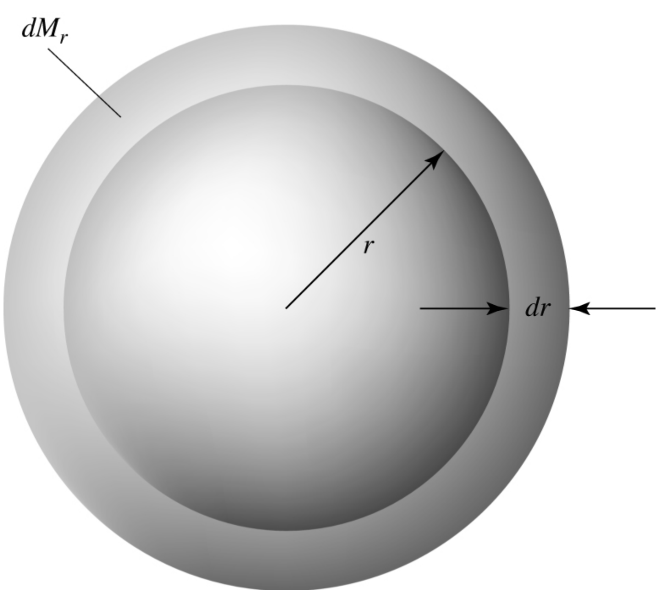
$$\boxed{\frac{dP}{dr} = -G\frac{M_r \rho}{r^2} = -\rho g,}$$

where $g \equiv GM_r/r^2$ is the local acceleration of gravity at radius r.

This is the condition for **hydrostatic equilibrium**, the first of fundamental equations of stellar structure.

For a star to be static, a pressure gradient dP/dr must exist to counteract the force of gravity; furthermore, P decreases with increasing radius.

Equation of Mass Conservation



A shell of mass dM_r

volume $dV = 4\pi\, r^2\, dr$

$$dM_r = \rho(4\pi r^2 dr)$$

$$\boxed{\frac{dM_r}{dr} = 4\pi r^2 \rho,}$$

Mass conservation equation, the second of fundamental equations of stellar structure.

Pressure Equation of State

Ideal gas law: ***P V = N k T***, where P is the pressure, V is the volume, N is the number of particles, T is the temperature, and k is Boltzmann's constant.

Number density $n \equiv N / V$; therefore, $P_g = n k T$.

$$n = \frac{\rho}{\overline{m}} \qquad\qquad P_g = \frac{\rho k T}{\overline{m}}$$

where $\overline{m}$ is the average mass of a gas particle.

$$\mu \equiv \frac{\overline{m}}{m_H} \qquad\qquad \boxed{P_g = \frac{\rho k T}{\mu m_H}.}$$

μ is the **mean molecular weight**, the average mass of a free particle in the gas in units of the mass of hydrogen m_H.

$$m_H = 1.6735\times10^{-27}\ \text{kg}$$

The mean molecular weight depends on the composition of the gas as well as on the state of ionization of each species.

For a complete neutral gas

$$\overline{m}_n = \frac{\sum_j N_j m_j}{\sum_j N_j},$$

where N_j and m_j are number and mass of atoms of type j .

Dividing the equation by m_H , we get

$$\mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j},$$

where $A_j \equiv m_j/m_H$.

For a completely ionized gas,

$$\mu_i \simeq \frac{\sum_j N_j A_j}{\sum_j N_j (1 + z_j)},$$

where $1 + z_j$ accounts for the nucleus plus the number of free electrons resulting from complete ionization of an atom of type *j.*

$$\mu\, m_{\mathrm{H}} = \frac{total\ mass}{total\ number\ of\ particles}$$

For a neutral gas:

$$\frac{1}{\mu_n m_H} = \frac{\sum_j N_j}{\sum_j N_j m_j}$$

$$= \frac{\text{total number of particles}}{\text{total mass of gas}}$$

$$= \sum_j \frac{\text{number of particles from } j}{\text{mass of particles from } j} \cdot \frac{\text{mass of particles from } j}{\text{total mass of gas}}$$

$$= \sum_j \frac{N_j}{N_j A_j m_H} X_j$$

$$= \sum_j \frac{1}{A_j m_H} X_j,$$

where X_j is the mass fraction of atoms of type j .

$$\frac{1}{\mu_n} = \sum_j \frac{1}{A_j} X_j$$

$$\simeq X + \frac{1}{4} Y + \left\langle \frac{1}{A} \right\rangle_n Z$$

$\langle 1/A \rangle_n$ is a weighted average of all elements in the gas heavier than helium. For solar abundance, $\langle 1/A \rangle_n \sim 1/15.5$.

For a completely ionized gas:

$$\frac{1}{\mu_i} = \sum_j \frac{1+z_j}{A_j} X_j$$
$$\simeq 2X + \frac{3}{4}Y + \left\langle \frac{1+z}{A} \right\rangle_i Z$$

Heavy elements usually have the same number of protons and neutrons in their nuclei, so $A_j \sim 2\, z_j$ and

$$\left\langle \frac{1+z}{A} \right\rangle_i \simeq \frac{1}{2}$$

For a typical composition of a young star, $X = 0.70$, $Y = 0.28$, and $Z = 0.02$, $\mu_n = 1.30$, and $\mu_i = 0.62$.

Average kinetic energy per particle

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$$

The factor of 3 arose from the three coordinate directions (or degrees of freedom). The average kinetic energy per degree of freedom is ½kT.

Fermi-Dirac and Bose-Einstein Statistics

Fermions, such as electrons, protons, and neutrons, obey the Heisenberg uncertainty principle and the Pauli exclusion principle. The Fermi-Dirac distribution function of Fermions considers these principles, and leads to a very different pressure equation of state for extremely dense matter, such as the interiors of white dwarfs and neutron stars.

Bosons, such as photons, do not obey Pauli exclusion principle. They obey Bose-Einstein statistics.

In the classical limit without extremely high densities, the Fermi-Dirac and Bose-Einstein distribution functions are similar to the Maxwell-Boltzmann distribution function.

The Contribution due to Radiation Pressure

From Lecture 10 (Sep 18):

For blackbody radiation:

$$P_{\rm rad} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\,d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4 = \frac{1}{3}u$$

The blackbody radiation pressure is 1/3 the energy density.

The total pressure consists of gas pressure and radiation pressure:

$$P_t = \frac{\rho k T}{\mu m_H} + \frac{1}{3}aT^4$$

At the center of the Sun, $T_c \sim 1.57\times10^7$ K, $\mu \sim 0.62$. Using the average density of the Sun (1410 kg m^{-3}), the gas pressure is $\sim 2.9\times10^{14}$ N m^{-2}. The radiation pressure is $\sim 1.53\times10^{13}$ N m^{-2}, much smaller than the gas pressure.