Astronomy 404
September 25, 2013

Chapter 9. Stellar Atmospheres

The Radiative Transfer Equation

Emission increases the specific intensity:

$$dI_\lambda = j_\lambda \rho \, ds$$

where j_λ **is the emission coefficient** of the gas, in units of **m s^{-3} sr^{-1}**.

Now consider both the absorption and emission,

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda \, ds + j_\lambda \rho \, ds$$

Dividing both side by $-\kappa_\lambda \rho \, ds$, we get

$$-\frac{1}{\kappa_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}$$

The equation of radiative transfer:

$$\boxed{-\frac{1}{\kappa_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - S_\lambda .}$$

where $S_\lambda \equiv j_\lambda / \kappa_\lambda$ is called the **source function**, in W m^{-3}sr^{-1}.

If $I_\lambda < S_\lambda$, $dI_\lambda/ds > 0$ I_λ increases
If $I_\lambda > S_\lambda$, $dI_\lambda/ds < 0$ I_λ decreases
The intensity of light tends to become equal to the local source function, which may vary too rapidly for the intensity to catch up…

If there is thermodynamic equilibrium, there is no energy flow, so

$dI_\lambda/ds = 0$ and $I_\lambda = B_\lambda$ $\rightarrow$ $S_\lambda = B_\lambda$

For the case of thermodynamic equilibrium, the source function is equal to the Planck function, $S_\lambda = B_\lambda$.

Assuming LTE, S_λ can be replaced by B_λ .

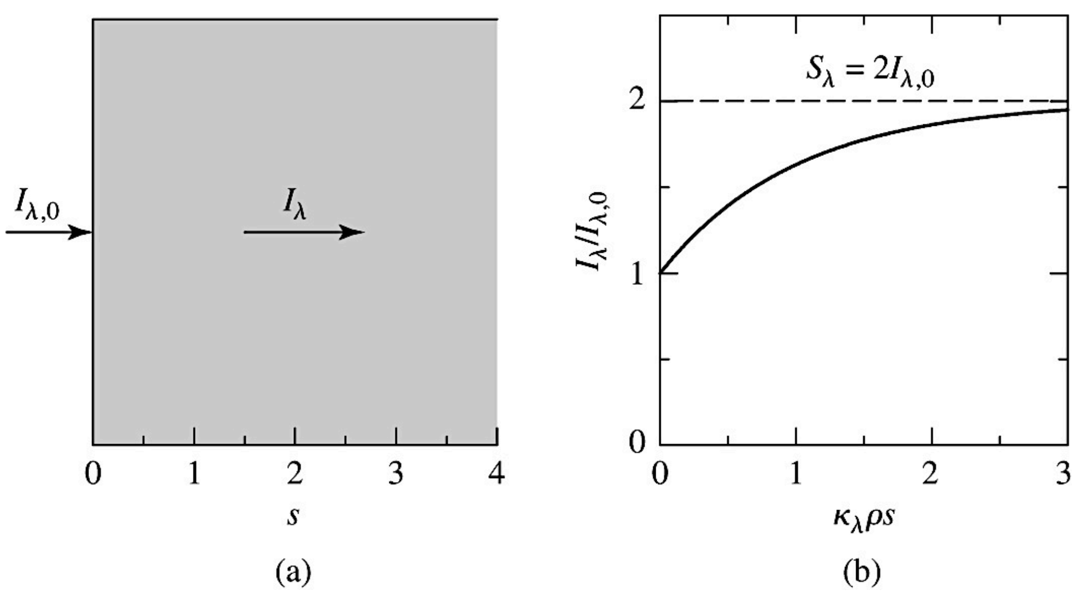
$$-\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - S_\lambda.$$

Assuming constant ρ, constant κ_λ, and constant S_λ , the transfer equation can be solved:

$$I_\lambda(s) = I_{\lambda,0}\, e^{-\kappa_\lambda \rho s} + S_\lambda(1 - e^{-\kappa_\lambda \rho s})$$

If $S_\lambda = 2\, I_{\lambda,0}$, I_λ converges to S_λ quickly.

(Recall that $1/\kappa_\lambda\rho$ is the mean free path of a photon and that $\tau_\lambda = \kappa_\lambda \rho s$.)



Plane-Parallel Atmosphere

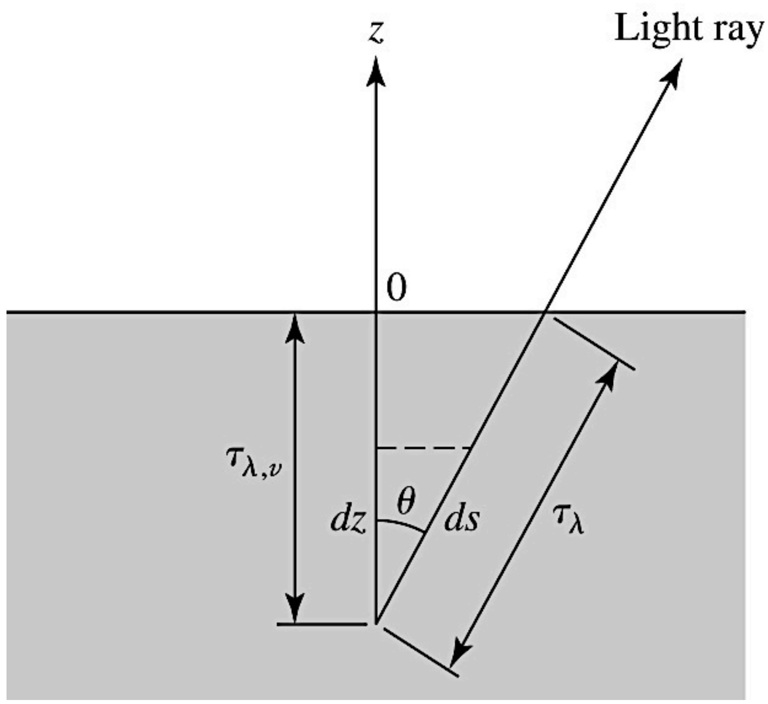
Since $d\tau_\lambda = \kappa_\lambda \rho \, ds$, the equation of radiative transfer can be written as:

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

Neither τ_λ nor s corresponds to a unique geometrical depth in the atmosphere. We can define τ_λ as a function of stellar radius, but as stellar atmosphere is thin compared with the stellar radius, we can consider stellar atmosphere as *a plane-parallel slab*.

The z-axis is in the vertical direction, with $z = 0$ at the top of this plane-parallel atmosphere. The vertical optical depth, $\tau_{\lambda,v}(z)$, is defined as:

$$\tau_{\lambda,v}(z) \equiv \int_z^0 \kappa_\lambda \rho \, dz$$



$$dz = ds\cos\theta, \quad \text{and} \quad \tau_{\lambda,v}(z) \equiv \int_z^0 \kappa_\lambda \rho\, dz$$

$$\tau_\lambda = \frac{\tau_{\lambda,v}}{\cos\theta} = \tau_{\lambda,v}\sec\theta$$

The transfer equation for plane-parallel atmosphere can be written as :

$$\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda$$

If we assume the opacity is independent of wavelength (*called grey atmosphere*), or use the Rosseland mean opacity $\overline{\kappa}$, then we have

$$I = \int_0^\infty I_\lambda\, d\lambda \quad \text{and} \quad S = \int_0^\infty S_\lambda\, d\lambda$$

and the transfer equation becomes:

$$\cos\theta \frac{dI}{d\tau_v} = I - S$$

Two useful relations can be derived from this equation.

First, we can integrate this equation over all solid angle:

$$\frac{d}{d\tau_v}\int I\cos\theta\, d\Omega = \int I\, d\Omega - S\int d\Omega$$

$$\langle I_\lambda\rangle \equiv \frac{1}{4\pi}\int I_\lambda\, d\Omega$$ and $$F_\lambda\, d\lambda = \int I_\lambda\, d\lambda\, \cos\theta\, d\Omega$$

$$\frac{dF_{\rm rad}}{d\tau_v} = 4\pi(\langle I\rangle - S)$$

Second, we can multiple the equation by cos θ then integrate over Ω :

$$\frac{d}{d\tau_v}\int I\cos^2\theta\, d\Omega = \int I\cos\theta\, d\Omega - S\int \cos\theta\, d\Omega$$

Recall that

$$P_{\rm rad,\lambda}\, d\lambda = \frac{1}{c}\int_{\rm sphere} I_\lambda\, d\lambda\, \cos^2\theta\, d\Omega \qquad F_\lambda\, d\lambda = \int I_\lambda\, d\lambda\, \cos\theta\, d\Omega$$

$$\int \cos\theta\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \cos\theta\sin\theta\, d\theta\, d\phi = 0$$

$$\frac{dP_{\rm rad}}{d\tau_v} = \frac{1}{c}F_{\rm rad} \qquad \frac{dP_{\rm rad}}{dr} = -\frac{\overline{\kappa}\rho}{c}F_{\rm rad}$$

The gradient in radiation pressure drives the flow of photons!

In an equilibrium stellar atmosphere, every process of absorption is balanced by an inverse process of emission. No energy is added or subtracted from the radiation field.

$F_{\rm rad} = \text{constant} = F_{\rm surf} = \sigma T_e^4$ and $dF_{\rm rad}/d\tau_v = 0$

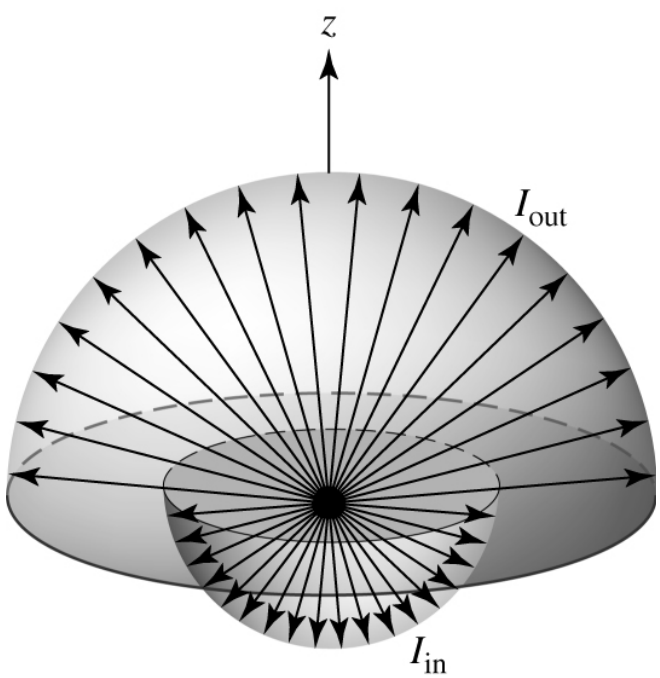
$\langle I \rangle = S$

$$\frac{dP_{\rm rad}}{d\tau_v} = \frac{1}{c} F_{\rm rad} \longrightarrow P_{\rm rad} = \frac{1}{c} F_{\rm rad} \tau_v + C$$

We can find radiation pressure as a function of optical depth.

The Eddington Approximation

The intensity is the $+z$ direction is $I_{\rm out}$, and in the $-z$ direction $I_{\rm in}$. $I_{\rm in} = 0$ at the top of the atmosphere, where $\tau_v = 0$.



$$\langle I \rangle = \frac{1}{2}(I_{\text{out}} + I_{\text{in}})$$

$$F_{\text{rad}} = \pi\,(I_{\text{out}} - I_{\text{in}})$$

$$P_{\text{rad}} = \frac{2\pi}{3c}(I_{\text{out}} + I_{\text{in}}) = \frac{4\pi}{3c}\langle I \rangle$$

$$P_{\text{rad}} = \frac{1}{c} F_{\text{rad}} \tau_v + C$$

$$\frac{4\pi}{3c}\langle I \rangle = \frac{1}{c} F_{\text{rad}} \tau_v + C.$$

At the top of the atmosphere $\tau_v = 0$ and $I_{\text{in}} = 0$.

$$\langle I(\tau_v = 0) \rangle = F_{\text{rad}}/2\pi$$

$$C = \frac{2}{3c} F_{\text{rad}}$$

$$\langle I \rangle = \frac{3\sigma}{4\pi} T_e^4 \left(\tau_v + \frac{2}{3}\right)$$

where T_e is the effective temperature.

With LTE, $S_\lambda = B_\lambda$ and $S = B = \frac{\sigma T^4}{\pi}$. $\langle I \rangle = S = \frac{\sigma T^4}{\pi}$.

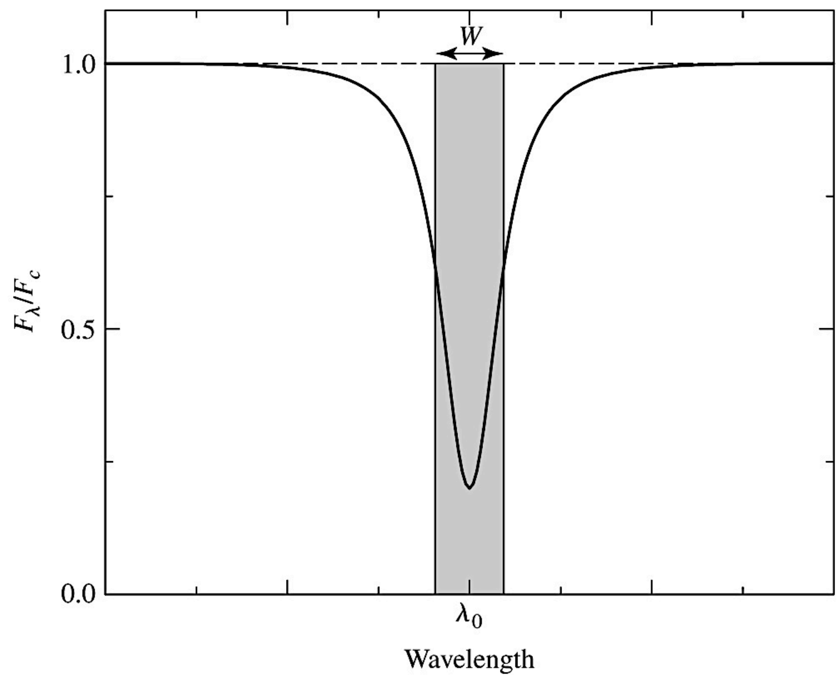
$$\boxed{T^4 = \frac{3}{4} T_e^4 \left(\tau_v + \frac{2}{3}\right).}$$

$T = T_e$ at $\tau_v = 2/3$, not at $\tau_v = 0$. *When we look at a star, we see down to a vertical optical depth of ~ 2/3, averaged over the disk of the star.*

You can do a similar exercise to determine **limb darkening**. You can read **page 264-266** for your own entertainment.

The Profiles of Spectral Lines

To measure the strength of an absorption line, we normalize the spectrum, that is, making the continuum flux = 1, and integrate over the absorbed area.



The radiant flux is plotted. F_c is the continuum.
$(F_c - F_\lambda)/ F_c$ is the depth of a line.
The equivalent width of a line is defined as:

$$W = \int \frac{F_c - F_\lambda}{F_c} d\lambda$$

Full width at half-maximum = $(\Delta\lambda)_{1/2}$
Optically thick -- if line core reaches zero flux.
Optically thin -- if line core is not zero.

The core of a line is formed at higher regions of the atmosphere (cooler), and the wings of a line is formed deeper in the atmosphere (hotter), until it merges with the continuum forming region at an optical depth of 2/3.

Processes that Broaden Spectral Lines

Natural Broadening (Heisenberg's uncertainty principle)

$$\Delta E \approx \frac{\hbar}{\Delta t}$$

$$\Delta\lambda \approx \frac{\lambda^2}{2\pi c}\left(\frac{1}{\Delta t_i} + \frac{1}{\Delta t_f}\right)$$

where Δt_i and Δt_f are the lifetimes of an electron in its initial and final states, respectively.

$$(\Delta\lambda)_{1/2} = \frac{\lambda^2}{\pi c}\frac{1}{\Delta t_0}$$